

# HYPERBOLIC FIXED POINTS AND PERIODIC ORBITS OF HAMILTONIAN Diffeomorphisms

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**ABSTRACT.** We prove that for a certain class of closed monotone symplectic manifolds any Hamiltonian diffeomorphism with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. Among the manifolds in this class are complex projective spaces, some Grassmannians, and also certain product manifolds such as the product of a projective space with a symplectically aspherical manifold of low dimension.

A key to the proof of this theorem is the fact that the energy required for a Floer connecting trajectory to approach an iterated hyperbolic orbit and cross its fixed neighborhood is bounded away from zero by a constant independent of the order of iteration. This result, combined with certain properties of the quantum product specific to the above class of manifolds, implies the existence of infinitely many periodic orbits.

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## 1. INTRODUCTION AND THE MAIN RESULT

**1.1. Introduction.** The main result of the paper is that for a certain class of closed monotone symplectic manifolds any Hamiltonian diffeomorphism with a hyperbolic fixed or periodic point must necessarily have infinitely many periodic orbits. This class of manifolds includes complex projective spaces, some Grassmannians, and also certain product manifolds such as the product of a projective space with a symplectically aspherical manifold of low dimension.

To put this result in perspective, recall that, for many closed symplectic manifolds, every Hamiltonian diffeomorphism has infinitely many periodic orbits and, in fact, periodic orbits of arbitrarily large (prime) period whenever the fixed points are isolated. This unconditional existence of infinitely many periodic orbits is often referred to as the Conley conjecture. As of this writing, the Conley conjecture has been shown to hold for all symplectic manifolds  $M$  with  $c_1(TM)|_{\pi_2(M)} = 0$  and also for negative monotone manifolds; see [CGG, GG1, He] and also [FH, Gi2, GG3, Hi, LeC, SZ] for some important, but less general, results. Ultimately, one can expect the Conley conjecture to hold for most symplectic manifolds.

There are, however, notable exceptions. The simplest one is  $S^2$ : an irrational rotation of  $S^2$  about the  $z$ -axis has only two periodic orbits, which are also the fixed points; these are the poles. In fact, any manifold that admits a Hamiltonian torus action with isolated fixed points also admits a Hamiltonian diffeomorphism with finitely many periodic orbits. For instance, such a diffeomorphism is generated by a generic element of the torus. In particular, complex projective spaces, the Grassmannians, and, more generally, most of the coadjoint orbits of compact Lie groups as well as symplectic toric manifolds all admit Hamiltonian diffeomorphisms with finitely many periodic orbits.

An analogue of the Conley conjecture applicable to such manifolds is the conjecture that a Hamiltonian diffeomorphism with “more than necessary” fixed points has infinitely many periodic orbits. Here “more than necessary” is left deliberately vague although, from the authors’ perspective, it should be interpreted as a lower bound arising from some version of the Arnold conjecture. For  $\mathbb{CP}^n$ , the expected threshold is  $n + 1$ . This variant of the Conley conjecture is inspired by a celebrated theorem of Franks asserting that a Hamiltonian diffeomorphism of  $S^2$  with at least three fixed points must have infinitely many periodic orbits, [Fr1, Fr2]; see also [FH, LeC] for further refinements and [BH, CKRTZ, Ke] for symplectic topological proofs. (It is worth pointing out that Franks’ theorem holds for area preserving homeomorphisms. However, the discussion of possible generalizations of this stronger result to higher dimensions is far outside the scope of this paper.)

There are few results directly supporting this conjecture in dimensions greater than two. Some evidence is provided by the results of [Gü], where a “local version” of the conjecture is considered. The main theorem of the present paper can also be viewed as supporting the conjecture. In fact, the conjecture, at least for non-degenerate Hamiltonian diffeomorphisms of  $\mathbb{CP}^n$  and some other manifolds, would follow if we could replace a hyperbolic fixed point by a non-elliptic fixed point in the main theorem. (Of course, our main result implies the non-degenerate case of Franks’ theorem, but in general it appears to add little to what is already known in dimension two.)

Hyperbolicity is central to the proof of the theorem. The argument hinges on a result, perhaps of independent interest, asserting that the energy required for a

Floer connecting trajectory of an iterated Hamiltonian to approach a hyperbolic orbit and cross its fixed neighborhood cannot be arbitrarily small: it is bounded away from zero by a constant independent of the order of iteration. This is an exclusive feature of hyperbolic fixed points.

**1.2. Main theorem.** Consider a closed monotone symplectic manifold  $M^{2n}$  with minimal Chern number  $N$ . We denote by  $HQ_*(M)$  the quantum homology ring of  $M$ , for the sake of simplicity taken over  $\mathbb{F} = \mathbb{Z}_2$ , and by  $q$  the generator of the Novikov ring  $\Lambda = \mathbb{F}[q, q^{-1}]$ , normalized to have degree  $|q| = -2N$ . We refer the reader to Section 2 for a detailed discussion of our conventions and notation.

The main result of the paper is

**Theorem 1.1.** *Let  $M$  be a closed strictly monotone symplectic manifold of dimension  $2n$  (i.e.,  $M$  is monotone and  $c_1(TM)|_{\pi_2(M)} \neq 0$  and  $[\omega]|_{\pi_2(M)} \neq 0$ ) such that  $N \geq n/2 + 1$ . Assume that*

$$\beta * \alpha = q^\nu[M] \tag{1.1}$$

*in  $HQ_*(M)$  for some ordinary homology classes  $\alpha \in H_*(M)$  and  $\beta \in H_*(M)$  with  $|\alpha| < 2n$  and  $|\beta| < 2n$ , and that*

- (i)  $\nu = 1$  or
- (ii)  $|\alpha| \geq 3n + 1 - 2N$ .

*Then any Hamiltonian diffeomorphism  $\varphi_H$  of  $M$  with a contractible hyperbolic periodic orbit  $\gamma$  has infinitely many periodic orbits.*

**Remark 1.2** (Non-contractible Orbits). The contractibility assumption on the orbit  $\gamma$  can be dropped once the monotonicity requirement is suitably adjusted. To be more precise, assume that  $M$  is toroidally monotone in addition to the conditions of the theorem. (See Section 2.4 for the definition.) Then any Hamiltonian diffeomorphism of  $M$  with a hyperbolic periodic orbit  $\gamma$  (not necessarily contractible) has infinitely many periodic orbits. These orbits lie in the collection of free homotopy classes formed by the iterations of  $\gamma$ . The proof of this fact is essentially identical to the proof of the theorem; see Remark 4.4.

Among the manifolds meeting the requirements of Theorem 1.1 are complex projective spaces  $\mathbb{CP}^n$ , complex Grassmannians  $\text{Gr}(2, N)$ ,  $\text{Gr}(3, 6)$  and  $\text{Gr}(3, 7)$ , the products  $\mathbb{CP}^m \times P^{2d}$  and  $\text{Gr}(2, N) \times P^{2d}$  where  $P$  is symplectically aspherical and  $d + 2 \leq m$  in the former case and  $d \leq 2$  in the latter, and monotone products  $\mathbb{CP}^m \times \mathbb{CP}^m$ . These manifolds also meet the requirements of the non-contractible version of the theorem (Remark 1.2), provided that the products are toroidally monotone. (We do not have any example of a manifold with  $N \geq n/2 + 1$  satisfying the conditions of Case (ii) of the theorem, but not of Case (i).)

**Remark 1.3.** We emphasize that in Theorem 1.1 we do not make any non-degeneracy assumptions on  $\varphi_H$ . Note also that, in contrast with the Conley conjecture type results discussed above, we do not claim the existence of periodic orbits with arbitrarily large period.

**1.3. Discussion.** As was mentioned in the introduction, a key to the proof of Theorem 1.1 is a lower bound  $c_\infty > 0$ , independent of the order of iterations, on the energy that a Floer trajectory of an iterated Hamiltonian requires to approach a hyperbolic orbit and cross its fixed neighborhood. (The assumption that the orbit is hyperbolic is essential here; see Remark 3.4.) This is Theorem 3.1, which is a

purely local result. Its proof relies on the Gromov compactness theorem in the form established in [Fi].

It is quite likely that the hypotheses of Theorem 1.1 can be relaxed via, for instance, further refining Theorem 3.1. (See also Remark 4.3.) Hypothetically, a result similar to Theorem 1.1 should hold for monotone manifolds without any restrictions on the minimal Chern number  $N$  or even for weakly monotone and rational symplectic manifolds. Moreover, we expect the periodic orbits to enter an arbitrarily small neighborhood of the hyperbolic orbit. Note, however, that the algebraic requirement (1.1), or some variant of it, is central to the proof.

In the context of Hamiltonian dynamical systems, the presence of one hyperbolic orbit implies,  $C^1$ -generically, the existence of transverse homoclinic points via the so-called connecting lemma; see [Ha, Xi]. (The genericity assumption is essential here, although hypothetically this could be a  $C^\infty$ -generic condition rather than  $C^1$ .) The existence of transverse homoclinic points has, in turn, rich dynamical consequences among which is the existence of infinitely many periodic orbits; see, e.g., [Ru, Ze] and references therein. Thus, under certain additional conditions on the ambient manifold, Theorem 1.1 recovers a fraction of this dynamics, but does this unconditionally rather than generically. Note also that the existence of infinitely many periodic orbits is a  $C^1$ -generic phenomenon, as follows from the closing lemma (see [PR]), and in many instances even  $C^\infty$ -generic (see [GG2]).

Finally, note that one can expect an analogue of Theorem 3.1 to hold for hyperbolic periodic orbits of Reeb flows and have applications, similar to Theorem 1.1, to the existence of infinitely many closed Reeb orbits; we will consider this circle of questions elsewhere.

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## 2. PRELIMINARIES

The goal of this section is to set notation and conventions, following mainly [GG1], and to give a brief review of Floer homology and several other notions used in the paper.

**2.1. Conventions and notation.** Let  $(M^{2n}, \omega)$  be a closed symplectic manifold, which in this paper (except Section 3) is assumed to be *monotone*, i.e.,  $[\omega] \mid_{\pi_2(M)} = \lambda c_1(TM) \mid_{\pi_2(M)}$  for some  $\lambda \geq 0$ . Furthermore, we focus on the case where  $\lambda \neq 0$  and  $c_1(TM) \mid_{\pi_2(M)} \neq 0$  and refer to such manifolds as *strictly monotone*. (Note a somewhat unconventional position of the *monotonicity constant*  $\lambda$  in the above definition.) The positive generator  $\lambda_0$  of the group  $\langle [\omega], \pi_2(M) \rangle \subset \mathbb{R}$  formed by the integrals of  $\omega$  over the spheres is called the *rationality constant*. Likewise, the *minimal Chern number*  $N$  is the positive generator of the group  $\langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z}$ . Clearly,  $\lambda_0 = \lambda N$ .

All Hamiltonians  $H$  considered in this paper are assumed to be  $\kappa$ -periodic in time (i.e.,  $H$  is a function  $S_\kappa^1 \times M \rightarrow \mathbb{R}$ , where  $S_\kappa^1 = \mathbb{R}/\kappa\mathbb{Z}$ ) and the period  $\kappa$  is

always a positive integer. When the period  $\kappa$  is not specified, it is equal to one, and  $S^1 = \mathbb{R}/\mathbb{Z}$ . We set  $H_t = H(t, \cdot)$  for  $t \in S_\kappa^1$ . The Hamiltonian vector field  $X_H$  of  $H$  is defined by  $i_{X_H}\omega = -dH$ . The (time-dependent) flow of  $X_H$  will be denoted by  $\varphi_H^t$  and its time-one map by  $\varphi_H$ . Such time-one maps are referred to as *Hamiltonian diffeomorphisms*. A one-periodic Hamiltonian  $H$  can also be treated as  $\kappa$ -periodic. In this case, we will use the notation  $H^{\natural\kappa}$  and, abusing terminology, call  $H^{\natural\kappa}$  the  $\kappa$ th iteration of  $H$ .

Let  $K$  and  $H$  be one-periodic Hamiltonians such that  $K_1 = H_0$  and  $H_1 = K_0$ . We denote by  $K \natural H$  the two-periodic Hamiltonian equal to  $K_t$  for  $t \in [0, 1]$  and  $H_{t-1}$  for  $t \in [1, 2]$ . Thus  $H^{\natural\kappa} = H \natural \dots \natural H$  ( $\kappa$  times).

Let  $x: S_\kappa^1 \rightarrow M$  be a contractible loop. A *capping* of  $x$  is a map  $u: D^2 \rightarrow M$  such that  $u|_{S_\kappa^1} = x$ . Two cappings  $u$  and  $v$  of  $x$  are considered to be equivalent if the integral of  $c_1(TM)$  (and hence of  $\omega$ ) over the sphere obtained by attaching  $u$  to  $v$  is equal to zero. A capped closed curve  $\bar{x}$  is, by definition, a closed curve  $x$  equipped with an equivalence class of cappings. In what follows, the presence of capping is always indicated by the bar. We denote the collection of capped one-periodic orbits of  $H$  by  $\bar{\mathcal{P}}(H)$ .

The action of a one-periodic Hamiltonian  $H$  on a capped loop  $\bar{x} = (x, u)$  is defined by

$$\mathcal{A}_H(\bar{x}) = - \int_u \omega + \int_{S^1} H_t(x(t)) dt.$$

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of  $\mathcal{A}_H$  on this covering space are exactly capped one-periodic orbits of  $X_H$ . The *action spectrum*  $\mathcal{S}(H)$  of  $H$  is the set of critical values of  $\mathcal{A}_H$ . This is a zero measure set; see, e.g., [HZ]. When  $M$  is monotone as we are assuming here,  $\mathcal{S}(H)$  is closed, and hence nowhere dense. These definitions extend to  $\kappa$ -periodic orbits and Hamiltonians in the obvious way. Clearly, the action functional is homogeneous with respect to iteration:

$$\mathcal{A}_{H^{\natural\kappa}}(\bar{x}^\kappa) = \kappa \mathcal{A}_H(\bar{x}). \quad (2.1)$$

Here  $\bar{x}^\kappa$  stands for the  $\kappa$ th iteration of the capped orbit  $\bar{x}$ .

Throughout most of the paper, a *periodic orbit is assumed to be contractible, unless explicitly stated otherwise*. More specifically, we consider non-contractible orbits in Remarks 1.2 and 4.4, and in Sections 2.4 and 3.

A periodic orbit  $x$  of  $H$  is said to be *non-degenerate* if the linearized return map  $d\varphi_H: T_{x(0)}M \rightarrow T_{x(0)}M$  has no eigenvalues equal to one, and  $x$  is called *hyperbolic* if none of the eigenvalues has absolute value one. A Hamiltonian is non-degenerate if all its one-periodic orbits are non-degenerate.

Let  $\bar{x}$  be a non-degenerate (capped) periodic orbit. The *Conley–Zehnder index*  $\mu_{\text{CZ}}(\bar{x}) \in \mathbb{Z}$  is defined, up to a sign, as in [Sa, SZ]. More specifically, in this paper, the Conley–Zehnder index is the negative of that in [Sa]. In other words, we normalize  $\mu_{\text{CZ}}$  so that  $\mu_{\text{CZ}}(\bar{x}) = n$  when  $x$  is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. The *mean index*  $\Delta_H(\bar{x}) \in \mathbb{R}$  measures, roughly speaking, the total angle swept by certain eigenvalues with absolute value one of the linearized flow  $d\varphi_H^t$  along  $x$  with respect to the trivialization associated with the capping; see [Lo, SZ]. (Sometimes we use the notation  $\Delta(\bar{x})$  when the Hamiltonian is clear from the context.) The mean index is defined regardless of whether  $x$  is degenerate or not, and  $\Delta_H(\bar{x})$  depends

continuously on  $H$  and  $\bar{x}$  in the obvious sense. When  $x$  is non-degenerate, we have

$$0 < |\Delta_H(\bar{x}) - \mu_{\text{CZ}}(H, \bar{x})| < n. \quad (2.2)$$

Furthermore, the mean index is also homogeneous with respect to iteration:

$$\Delta_{H^{\natural\kappa}}(\bar{x}^\kappa) = \kappa \Delta_H(\bar{x}). \quad (2.3)$$

We let  $\mathcal{A}_H(x) \in S_{\lambda_0}^1 = \mathbb{R}/\lambda_0\mathbb{Z}$  be the action  $\mathcal{A}_H(\bar{x})$  taken modulo  $\lambda_0$ , and hence independent of the capping. (The mean index  $\Delta_H(x) \in S_{2N}^1 = \mathbb{R}/2N\mathbb{Z}$  can be defined in a similar fashion.) Finally, recall that the *augmented action* of a one-periodic (or  $\kappa$ -periodic) orbit  $x$  is the difference

$$\tilde{\mathcal{A}}_H(x) = \mathcal{A}_H(\bar{x}) - \frac{\lambda}{2} \Delta_H(\bar{x}), \quad (2.4)$$

where on the right we use an arbitrary capping of  $x$ ; see [GG1]. By (2.7) and (2.8), the augmented action is independent of the capping and, by (2.1) and (2.3), homogeneous:

$$\tilde{\mathcal{A}}_{H^{\natural\kappa}}(x^\kappa) = \kappa \tilde{\mathcal{A}}_H(x). \quad (2.5)$$

## 2.2. Floer and quantum (co)homology.

**2.2.1. Floer homology.** In this subsection, we very briefly recall, mainly to set notation, the construction of the filtered Floer homology for strictly monotone symplectic manifolds, i.e., monotone manifolds with  $\lambda \neq 0$  and  $c_1(TM)|_{\pi_2(M)} \neq 0$ . We refer the reader to, e.g., [Fl, HS, MS, Sa, SZ] and for detailed accounts of the construction and for additional references.

Throughout this paper, all complexes and homology groups are, for the sake of simplicity, defined over the ground field  $\mathbb{F} = \mathbb{Z}_2$ . Let  $H$  be a one-periodic non-degenerate Hamiltonian on  $M$ . Denote by  $\text{CF}_k^{(-\infty, b)}(H)$ , where  $b \in (-\infty, \infty]$  is not in  $\mathcal{S}(H)$ , the vector space of finite formal sums

$$\alpha = \sum_{\bar{x} \in \mathcal{P}(H)} \alpha_{\bar{x}} \bar{x}.$$

Here  $\alpha_{\bar{x}} \in \mathbb{F}$  and  $|\bar{x}| := \mu_{\text{CZ}}(\bar{x}) + n = k$  and  $\mathcal{A}_H(\bar{x}) < b$ . (Since  $M$  is strictly monotone there is no need to consider semi-infinite sums.) The graded  $\mathbb{F}$ -vector space  $\text{CF}_*^{(-\infty, b)}(H)$  is endowed with the Floer differential  $\partial$  counting the anti-gradient trajectories of the action functional.

More specifically,  $\partial$  is defined as follows. Fix a generic (one-periodic in time) almost complex structure  $J$  compatible with  $\omega$  and consider solutions  $u: \mathbb{R} \times S^1 \rightarrow M$  of the Floer equation

$$\partial_s u + J \partial_t u = -\nabla H. \quad (2.6)$$

Here the gradient on the right is taken with respect to the one-periodic in time metric  $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$  on  $M$ , and  $(s, t)$  are the coordinates on  $\mathbb{R} \times S^1$ . Denote by  $\mathcal{M}(\bar{x}, \bar{y})$  the space of solutions of the Floer equation, (2.6), asymptotic to  $\bar{x}$  at  $-\infty$  and  $\bar{y}$  at  $\infty$  and such that the capping of  $\bar{y}$  is equivalent to the capping obtained by attaching  $u$  to the capping of  $\bar{x}$ . The space  $\mathcal{M}(\bar{x}, \bar{y})$  carries a natural free  $\mathbb{R}$ -action by shifts along the  $s$ -axis. Set  $\widehat{\mathcal{M}}(\bar{x}, \bar{y}) = \mathcal{M}(\bar{x}, \bar{y})/\mathbb{R}$ . For a generic choice of  $J$ , we have  $\dim \widehat{\mathcal{M}}(\bar{x}, \bar{y}) = |\bar{x}| - |\bar{y}| - 1$ . Let  $m(\bar{x}, \bar{y}) \in \mathbb{F}$  stand for the parity of  $\#\widehat{\mathcal{M}}(\bar{x}, \bar{y})$

when  $\widehat{\mathcal{M}}(\bar{x}, \bar{y})$  is zero-dimensional (one can show that then it is finite) and zero otherwise. Finally, set

$$\partial \bar{x} = \sum_{\bar{y}} m(\bar{x}, \bar{y}) \bar{y}.$$

It is well known that  $\partial^2 = 0$ ; see the references above.

The complexes  $\mathrm{CF}_*^{(-\infty, b)}(H)$  equipped with the differential  $\partial$  form a filtration of the total Floer complex  $\mathrm{CF}_*(H) := \mathrm{CF}_*^{(-\infty, \infty)}(H)$ . We set  $\mathrm{CF}_*^{(c, c')}(H) := \mathrm{CF}_*^{(-\infty, c')}(H) / \mathrm{CF}_*^{(-\infty, c)}(H)$ , where  $c$  and  $c'$  are not in  $\mathcal{S}(H)$  and  $-\infty \leq c < c' \leq \infty$ . The homology groups of these complexes are called the *filtered Floer homology* of  $H$  and denoted by  $\mathrm{HF}_*^{(c, c')}(H)$  or by  $\mathrm{HF}_*(H)$  when  $(c, c') = (-\infty, \infty)$ . These groups are independent of the choice of  $J$ . Every  $\mathbb{F}$ -vector space  $\mathrm{CF}_k(H)$  is finite-dimensional since  $M$  is strictly monotone.

Recall also that for any  $u \in \mathcal{M}(\bar{x}, \bar{y})$  we have

$$\mathcal{A}_H(\bar{x}) - \mathcal{A}_H(\bar{y}) = E(u), \text{ where } E(u) := \int_{\mathbb{R} \times S^1} \|\partial u\|^2 ds dt,$$

and that  $E(u)$  is referred to as the *energy* of  $u$ .

The total Floer complex and homology are modules over the *Novikov ring*  $\Lambda$  whose action on the complex is essentially by recapping the orbits. To define this ring, let us first set some notation.

Let  $\omega(A)$  and  $\langle c_1(TM), A \rangle$  denote the integrals of  $\omega$  and, respectively,  $c_1(TM)$  over a cycle  $A$ . Set

$$I_\omega(A) = -\omega(A) \text{ and } I_{c_1}(A) = -2 \langle c_1(TM), A \rangle,$$

where  $A \in \pi_2(M)$ . Thus  $I_\omega = (\lambda/2)I_{c_1}$  since  $M$  is assumed to be monotone. Let  $\Gamma = \pi_2(M)/\ker I_\omega = \pi_2(M)/\ker I_{c_1}$ . In other words,  $\Gamma$  is the quotient of  $\pi_2(M)$  by the equivalence relation where two spheres  $A$  and  $A'$  are considered to be equivalent if  $\langle c_1(TM), A \rangle = \langle c_1(TM), A' \rangle$  or, equivalently,  $\omega(A) = \omega(A')$ . Clearly  $\Gamma \simeq \mathbb{Z}$ , and the homomorphisms  $I_\omega$  and  $I_{c_1}$  descend to  $\Gamma$  from  $\pi_2(M)$ .

The group  $\Gamma$  acts on  $\mathrm{CF}_*(H)$  and on  $\mathrm{HF}_*(H)$  via recapping: an element  $A \in \Gamma$  acts on a capped one-periodic orbit  $\bar{x}$  of  $H$  by attaching the sphere  $A$  to the original capping. Denoting the resulting capped orbit by  $\bar{x} \# A$ , we have

$$\mu_{CZ}(\bar{x} \# A) = \mu_{CZ}(\bar{x}) + I_{c_1}(A) \text{ and } \mathcal{A}_H(\bar{x} \# A) = \mathcal{A}_H(\bar{x}) + I_\omega(A). \quad (2.7)$$

In a similar vein,

$$\Delta_H(\bar{x} \# A) = \Delta_H(\bar{x}) + I_{c_1}(A) \quad (2.8)$$

regardless of whether  $x$  is non-degenerate or not.

Since  $M$  is strictly monotone, the Novikov ring  $\Lambda$  may simply be taken to be the group algebra  $\mathbb{F}[\Gamma]$  of  $\Gamma$  over  $\mathbb{F}$ , i.e., the  $\mathbb{F}$ -algebra of formal finite linear combinations  $\sum \alpha_A e^A$ , where  $\alpha_A \in \mathbb{F}$ . The Novikov ring  $\Lambda$  is graded by setting  $|e^A| = I_{c_1}(A)$  for  $A \in \Gamma$ . The action of  $\Gamma$  turns  $\mathrm{CF}_*(H)$  and  $\mathrm{HF}_*(H)$  (but not  $\mathrm{CF}_*^{(c, c')}(H)$  or  $\mathrm{HF}_*^{(c, c')}(H)$ ) into  $\Lambda$ -modules. Recall that  $\Gamma \simeq \mathbb{Z}$  and denote by  $A_0$  the generator of  $\Gamma$  with  $I_{c_1}(A_0) = -2N$ . Setting  $q = e^{A_0} \in \Lambda$ , we have  $|q| = -2N$ , and the Novikov ring  $\Lambda$  is thus the ring of Laurent polynomials  $\mathbb{F}[q^{-1}, q]$ .

The definition of Floer homology extends to all, not necessarily non-degenerate, Hamiltonians by continuity. Let  $H$  be an arbitrary (one-periodic in time) Hamiltonian on  $M$  and let the end points  $c$  and  $c'$  of the action interval be outside  $\mathcal{S}(H)$ .



We set

$$\mathrm{HF}_*^{(c, c')}(H) = \mathrm{HF}_*^{(c, c')}(\tilde{H}),$$

where  $\tilde{H}$  is a non-degenerate, small perturbation of  $H$ . It is well known that the right hand side is independent of  $\tilde{H}$  as long as the latter is sufficiently close to  $H$ . Working with filtered Floer homology, *we always assume that the end points of the action interval are not in the action spectrum*. Finally, note that everywhere in this discussion we can replace one-periodic Hamiltonians by  $\kappa$ -periodic.

The total Floer homology is independent of the Hamiltonian and isomorphic to the homology of  $M$ . More precisely, we have

$$\mathrm{HF}_*(H) \cong H_*(M) \otimes \Lambda$$

as graded  $\Lambda$ -modules.

**2.2.2. Quantum homology.** The total Floer homology  $\mathrm{HF}_*(H)$ , equipped with the pair-of-pants product, is an algebra over the Novikov ring  $\Lambda$ . This algebra is isomorphic to the (small) *quantum homology*  $\mathrm{HQ}_*(M)$  of  $M$ ; see, e.g., [MS]. On the level of  $\Lambda$ -modules, we have

$$\mathrm{HQ}_*(M) = H_*(M) \otimes \Lambda \quad (2.9)$$

with tensor product grading. Thus  $|\alpha \otimes e^A| = |\alpha| + I_{c_1}(A)$ , where  $\alpha \in H_*(M)$  and  $A \in \Gamma$ . The isomorphism between  $\mathrm{HF}_*(H)$  and  $\mathrm{HQ}_*(M)$  is usually defined by using the PSS-homomorphism; see [PSS] or [MS, Us]. Alternatively, at least in the monotone case, it can be obtained from a homotopy of  $H$  to an autonomous  $C^2$ -small Hamiltonian, see [Fl] and, e.g., [HS, On].

The *quantum product*  $\alpha * \beta$  of two ordinary homology classes  $\alpha$  and  $\beta$  is defined as

$$\alpha * \beta = \sum_{A \in \Gamma} (\alpha * \beta)_A e^A, \quad (2.10)$$

where the class  $(\alpha * \beta)_A \in H_*(M)$  is determined via certain Gromov–Witten invariants of  $M$  and has degree  $|\alpha| + |\beta| - 2n - I_{c_1}(A)$ ; see [MS]. Thus  $|\alpha * \beta| = |\alpha| + |\beta| - 2n$ .

Note that  $(\alpha * \beta)_0 = \alpha \cap \beta$ , where  $\cap$  stands for the cap product and  $\alpha$  and  $\beta$  are ordinary homology classes. Recall also that in (2.10) it suffices to limit the summation to the negative cone  $I_\omega(A) \leq 0$ . In particular, under our assumptions on  $M$ , we can write

$$\alpha * \beta = \alpha \cap \beta + \sum_{k > 0} (\alpha * \beta)_k q^k,$$

where  $|(\alpha * \beta)_k| = |\alpha| + |\beta| - 2n + 2Nk$ . This sum is finite.

The product  $*$  is a  $\Lambda$ -linear, associative, graded-commutative product on  $\mathrm{HQ}_*(M)$ . The fundamental class  $[M]$  is the unit in the algebra  $\mathrm{HQ}_*(M)$ . Thus  $a\alpha = (a[M]) * \alpha$ , where  $a \in \Lambda$  and  $\alpha \in H_*(M)$ , and  $|a\alpha| = |a| + |\alpha|$ . By the very definition, the ordinary homology  $H_*(M)$  is canonically embedded in  $\mathrm{HQ}_*(M)$ .

The map  $I_\omega$  extends to  $\mathrm{HQ}_*(M)$  as

$$I_\omega(\alpha) = \max \{I_\omega(A) \mid \alpha_A \neq 0\} = \max \{-\lambda_0 k \mid \alpha_k \neq 0\},$$

where  $\alpha = \sum \alpha_A e^A = \sum \alpha_k q^k$ . We have

$$I_\omega(\alpha + \beta) \leq \max \{I_\omega(\alpha), I_\omega(\beta)\} \quad (2.11)$$

and

$$I_\omega(\alpha * \beta) \leq I_\omega(\alpha) + I_\omega(\beta). \quad (2.12)$$



*Example 2.1.* Let  $M = \mathbb{CP}^n$ . Then  $N = n + 1$  and  $\mathrm{HQ}_*(\mathbb{CP}^n)$  is the quotient of  $\mathbb{F}[u] \otimes \Lambda$ , where  $u$  is the generator of  $H_{2n-2}(\mathbb{CP}^n)$ , by the ideal generated by the relation  $u^{n+1} = q[M]$ . Thus  $u^k = u \cap \dots \cap u$  ( $k$  times) when  $0 \leq k \leq n$ , and  $[pt] * u = q[M]$ . For this and further examples of quantum homology calculations and relevant references, we refer the reader to, e.g., [MS].

**2.3. Cap product.** In what follows, it is convenient to view the product on the Floer or quantum homology from a slightly different angle – namely, as a “module structure” over  $\mathrm{HQ}_*(M)$  on the collection of filtered Floer homology groups. We refer to this structure, somewhat misleadingly, as the *cap product* even though no cohomology is explicitly in the picture.

Let us now describe the action of the quantum homology on the filtered Floer homology in more detail. Let  $H$  be a non-degenerate Hamiltonian and  $J$  be a generic almost complex structure. Let  $[\sigma] \in H_*(M)$ . Pick a generic cycle  $\sigma$  representing  $[\sigma]$  and denote by  $\mathcal{M}(\bar{x}, \bar{y}; \sigma)$  the moduli space of solutions of the Floer equation, (2.6), asymptotic to  $\bar{x}$  at  $-\infty$  and  $\bar{y}$  at  $\infty$  and such that  $u(0, 0) \in \sigma$ . For a generic choice of  $\sigma$ , we have  $\dim \mathcal{M}(\bar{x}, \bar{y}; \sigma) = |\bar{x}| - |\bar{y}| - \mathrm{codim}(\sigma)$ . Let  $m(\bar{x}, \bar{y}; \sigma) \in \mathbb{F}$  be the parity of  $\#\mathcal{M}(\bar{x}, \bar{y}; \sigma)$  when this moduli space is zero-dimensional (one can show that then it is finite) and zero otherwise. Set

$$\Phi_\sigma(\bar{x}) = \sum_{\bar{y}} m(\bar{x}, \bar{y}; \sigma) \bar{y}.$$

For any  $c$  and  $c'$  outside  $\mathcal{S}(H)$ , the map

$$\Phi_\sigma: \mathrm{CF}_*^{(c, c')}(H) \rightarrow \mathrm{CF}_{*-\mathrm{codim}(\sigma)}^{(c, c')}(H)$$

commutes with the Floer differential and descends to a map

$$\Phi_{[\sigma]}: \mathrm{HF}_*^{(c, c')}(H) \rightarrow \mathrm{HF}_{*-\mathrm{codim}(\sigma)}^{(c, c')}(H),$$

which is independent of the choice of the cycle  $\sigma$  representing  $[\sigma]$ . The analytical details of this construction and complete proofs can be found in, e.g., [LO], in much greater generality than is needed here. Clearly,

$$\Phi_{[M]} = id. \quad (2.13)$$

The action of the class  $\alpha = q^\nu[\sigma] \in \mathrm{HQ}_*(M)$  is induced by the map

$$\Phi_{q^\nu \sigma}(\bar{x}) := \sum_{\bar{y}} m(q^\nu \bar{x}, \bar{y}; \sigma) \bar{y}.$$

Here, as in Section 2.2.1,  $q = e^{A_0}$  where  $A_0$  is the generator of  $\Gamma$  with  $I_{c_1}(A_0) = -2N$ . It is routine to check that  $\Phi_{q^\nu \sigma} = q^\nu \Phi_{[\sigma]}$ . (This is a consequence of the fact that  $\mathcal{M}(q^\nu \bar{x}, \bar{y}; \sigma) = \mathcal{M}(\bar{x}, q^{-\nu} \bar{y}; \sigma)$ .) Note that now the action interval is shifted by  $I_\omega(\alpha)$ , i.e.,

$$\Phi_\alpha: \mathrm{HF}_*^{(c, c')}(H) \rightarrow \mathrm{HF}_{* - 2n + |\alpha|}^{(c, c') + I_\omega(\alpha)}(H), \quad (2.14)$$

where  $(c, c') + s$  stands for  $(c + s, c' + s)$ .

By linearity over  $\Lambda$ , we extend  $\Phi_\alpha$  to all  $\alpha \in \mathrm{HQ}_*(M)$  so that (2.14) still holds. The maps  $\Phi_\alpha$  are linear in  $\alpha$  once the shift in filtration is taken into account. Namely, let  $(a, b)$  be any interval such that  $a \geq c + \max\{I_\omega(\alpha), I_\omega(\beta)\}$  and  $b \geq c' + \max\{I_\omega(\alpha), I_\omega(\beta)\}$ . Then the quotient–inclusion maps  $\iota_\alpha$  and  $\iota_\beta$  from  $\mathrm{HF}_*^{(c, c') + I_\omega(\alpha)}(H)$  and, respectively,  $\mathrm{HF}_*^{(c, c') + I_\omega(\beta)}(H)$  to  $H_*^{(a, b)}(H)$  are defined; see, e.g., [Gi1, Example 3.3]. Clearly,  $a \geq c + I_\omega(\alpha + \beta)$  and  $b \geq c' + I_\omega(\alpha + \beta)$  by

(2.11), and we also have the map  $\iota_{\alpha+\beta}$  from the target space of  $\Phi_{\alpha+\beta}$  to  $\mathrm{HF}_*^{(a,b)}(H)$ . Additivity takes the form

$$\iota_{\alpha+\beta}\Phi_{\alpha+\beta} = \iota_{\alpha}\Phi_{\alpha} + \iota_{\beta}\Phi_{\beta}.$$

The maps  $\Phi_{\alpha}$ , for all  $\alpha \in \mathrm{HQ}_*(M)$ , fit together to form an action of the quantum homology on the collection of filtered Floer homology groups.

This action is also multiplicative. In other words, we have

$$\Phi_{\beta}\Phi_{\alpha} = \Phi_{\beta*\alpha}. \quad (2.15)$$

Note that here, as in the case of additivity, the maps on the two sides of the identity have, in general, different target spaces. Thus (2.15) should, more accurately, be understood as that for any interval  $(a, b)$  with  $a \geq c + I_{\omega}(\alpha) + I_{\omega}(\beta)$  and  $b \geq c' + I_{\omega}(\alpha) + I_{\omega}(\beta)$  the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{HF}_*^{(c,c')}(H) & \xrightarrow{\Phi_{\alpha}} & \mathrm{HF}_{*-2n+|\alpha|}^{(c,c')+I_{\omega}(\alpha)}(H) & \xrightarrow{\Phi_{\beta}} & \mathrm{HF}_{*-4n+|\alpha|+|\beta|}^{(c,c')+I_{\omega}(\alpha)+I_{\omega}(\beta)}(H) \\ & \searrow \Phi_{\beta*\alpha} & & & \downarrow \\ & & \mathrm{HF}_{*-2n+|\beta*\alpha|}^{(c,c')+I_{\omega}(\beta*\alpha)}(H) & \longrightarrow & \mathrm{HF}_{*-2n+|\beta*\alpha|}^{(a,b)}(H) \end{array} \quad (2.16)$$

Here the vertical arrow and the bottom horizontal arrow are again the natural quotient–inclusion maps whose existence is guaranteed by our choice of  $a$  and  $b$  and (2.12). Note that (2.15) can be thought of as a form of associativity of the product in quantum or Floer homology; (2.15) was essentially established in [LO] and [PSS]; see also [MS, Remark 12.3.3].

On the total Floer homology  $\mathrm{HF}_*(H) \cong \mathrm{HQ}_*(M)$ , the cap product coincides with the quantum or pair-of-pants product.

**2.4. Non-contractible orbits.** To deal with the case of non-contractible orbits, we need to strengthen the monotonicity requirement on  $M$ . Namely, we say that  $M$  is *toroidally monotone* if for every map  $v: \mathbb{T}^2 \rightarrow M$  we have  $\langle [\omega], [v] \rangle = \lambda \langle c_1(TM), [v] \rangle$  for some constant  $\lambda \geq 0$  independent of  $v$ . This condition is in general stronger than monotonicity, but weaker than requiring that  $[\omega] = \lambda c_1(TM)$ . (This can be easily seen by examining surfaces or products of surfaces.) Furthermore, assume from now on that  $M$  is in addition strictly monotone. Then the toroidal monotonicity constant  $\lambda$  is equal to the ordinary monotonicity constant and the minimal toroidal Chern number and the toroidal rationality constant (both defined in the obvious way) agree with their spherical counterparts.

Let  $\zeta$  be a free homotopy class of maps  $S^1 \rightarrow M$ . Fix a *reference loop*  $z \in \zeta$  and a symplectic trivialization of  $TM$  along  $z$ . (In fact, it would be sufficient to fix a trivialization of the canonical bundle of  $M$  along  $z$ .) A capped loop  $x$  is a loop in  $\zeta$  together with a cylinder (i.e., a homotopy) connecting it to  $z$ . Two cappings are considered equivalent when  $\langle c_1(TM), [v] \rangle = 0$ , where  $v: \mathbb{T}^2 \rightarrow M$  is the torus obtained by attaching the cappings to each other. (Due to the toroidal monotonicity condition, we also have  $\langle [\omega], [v] \rangle = 0$ .) As in the contractible case, we denote the capping by a bar.

Let now  $H$  be a Hamiltonian on  $M$ . We consider capped one-periodic (or  $\kappa$ -periodic) orbits of  $H$  in the class  $\zeta$ . For such orbits, the action  $\mathcal{A}_H(\bar{x})$  and the mean index  $\Delta_H(\bar{x})$  (and the Conley–Zehnder index  $\mu_{\mathrm{CZ}}(\bar{x})$  when  $x$  is non-degenerate) are

obviously well defined. Likewise, the augmented action (2.4) is still defined and independent of the capping, and (2.2) holds.

The construction of the filtered Floer complex  $\mathrm{CF}_*^{(c,c')}(H, \zeta)$  and the filtered Floer homology  $\mathrm{HF}_*^{(c,c')}(H, \zeta)$  goes through exactly as in the contractible case.

The Floer complex and homology are again modules over a Novikov ring  $\Lambda'$ . In general, this ring can be different from the spherical one. This is the case, for instance, when  $M$  is symplectically aspherical and toroidally monotone (e.g.,  $M = \mathbb{T}^2$ ) and  $\Lambda = \mathbb{F}$  while  $\Lambda' = \mathbb{F}[q, q^{-1}]$ . However, when  $M$  is both strictly monotone and toroidally monotone, as is assumed here, the natural inclusion  $\Lambda \rightarrow \Lambda'$  is an isomorphism and we can keep the notation  $\Lambda$  for both of the Novikov rings.

Since all one-periodic orbits of a  $C^2$ -small autonomous Hamiltonian are constant, and hence contractible, the total non-contractible Floer homology is trivial:  $\mathrm{HF}_*(H, \zeta) = 0$  whenever  $\zeta \neq 0$ . Furthermore, the pair-of-pants product is not defined on non-contractible filtered Floer homology for an individual class  $\zeta$ . (Such a product between  $\mathrm{HF}_*^{(c_1, c'_1)}(H, \zeta_1)$  and  $\mathrm{HF}_*^{(c_2, c'_2)}(K, \zeta_2)$  would take values in  $\mathrm{HF}_*^{(c_1+c_2, c'_1+c'_2)}(H \natural K, \zeta_1 + \zeta_2)$ .) However, the cap product with ordinary  $\mathrm{HQ}_*(M)$  is still defined and the discussion from Section 2.3 carries over word-for-word to the non-contractible case.

The action and the index of periodic orbits (and hence the grading and filtration of the Floer complex) do depend on the choice of  $z$  and, in the index case, on the trivialization of  $TM$  along  $z$ . Whenever we consider the iteration  $H^{\natural \kappa}$  of  $H$ , we simultaneously iterate the class  $\zeta$  and the reference curve  $z$  (i.e., pass to  $\kappa \cdot \zeta$  and  $z^\kappa$ ) and, in the obvious sense, also iterate the trivialization. Under these assumptions, the action, the mean index, and the augmented action are homogeneous with respect to the iterations, i.e., (2.1), (2.3), and (2.5) remain valid.

*Remark 2.2.* The index and action conventions of this section can be used even when  $\zeta = 0$ , resulting only in a shift of the standard grading and filtration in the Floer homology.

### 3. HYPERBOLIC FIXED POINTS

Our goal in this section is to establish a technical result underpinning the proof of Theorem 1.1. This is the fact that the energy required for a Floer connecting trajectory to approach an iterated hyperbolic orbit and cross its fixed neighborhood is bounded away from zero by a constant independent of the order of iteration.

**3.1. Setting.** Let us state precisely the conditions needed for this energy lower bound to hold. Let  $H$  be a one-periodic in time Hamiltonian on a symplectic manifold  $(M, \omega)$ , which is not required to be monotone or closed or satisfy any extra requirements at all; for the result we are interested in is essentially local. Fix an almost complex structure  $J$ , again one-periodic in time, compatible with  $\omega$ .

We consider solutions  $u: \Sigma \rightarrow M$  of the Floer equation (2.6), where  $\Sigma \subset \mathbb{R} \times S_\kappa^1$  is a closed domain (i.e., a closed subset with non-empty interior). Now, however, in contrast with Section 2.2.1, the period  $\kappa$  is not necessarily fixed, and the domain  $\Sigma$  of  $u$  need not be the entire cylinder  $\mathbb{R} \times S_\kappa^1$ . By definition, the energy of  $u$  is

$$E(u) = \int_{\Sigma} \|\partial_s u\|^2 ds dt.$$

Here  $\|\cdot\|$  stands for the norm with respect to  $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$ , and hence  $\|\cdot\|$  also depends on  $J$ .

Let  $\gamma$  be a hyperbolic (not necessarily contractible) one-periodic orbit of  $H$ . We say that  $u$  is asymptotic to  $\gamma^\kappa$ , the  $\kappa$ th iteration of  $\gamma$ , as  $s \rightarrow \infty$  if  $\Sigma$  contains some cylinder  $[s_0, \infty) \times S_\kappa^1$  and  $u(s, t) \rightarrow \gamma^\kappa(t)$   $C^\infty$ -uniformly in  $t$  as  $s \rightarrow \infty$ .

Finally, let  $U$  be a fixed (sufficiently small) closed neighborhood of  $\gamma$  with smooth boundary or, more precisely, such a neighborhood of the natural lift of  $\gamma$  to  $S^1 \times M$ .

**Theorem 3.1** (Ball-crossing Energy Theorem). *There exists a constant  $c_\infty > 0$ , independent of  $\kappa$  and  $\Sigma$ , such that for any solution  $u$  of the Floer equation, (2.6), with  $u(\partial\Sigma) \subset \partial U$  and  $\partial\Sigma \neq \emptyset$ , which is asymptotic to  $\gamma^\kappa$  as  $s \rightarrow \infty$ , we have*

$$E(u) > c_\infty. \quad (3.1)$$

Moreover, the constant  $c_\infty$  can be chosen to make (3.1) hold for all  $\kappa$ -periodic almost complex structures (with varying  $\kappa$ )  $C^\infty$ -close to  $J$  uniformly on  $\mathbb{R} \times U$ .

*Remark 3.2.* The most important point of this theorem is the fact that the energy required to approach  $\gamma^\kappa$  through  $U$  is bounded away from zero by a constant  $c_\infty$  independent of the iteration  $\kappa$  and also of the domain  $\Sigma$  of  $u$ . (Naturally,  $c_\infty$  depends on  $H$  and  $J$  and  $\omega$ .) A similar lower bound obviously holds for solutions of the Floer equation leaving  $\gamma^\kappa$  through  $U$ , i.e., asymptotic to  $\gamma^\kappa$  as  $s \rightarrow -\infty$ . Clearly, the requirement that the orbit  $\gamma$  is one-periodic can be replaced by the assumption that it is just a periodic orbit. Furthermore, in the “moreover” part of the theorem, we can take, for instance,  $\kappa$ -periodic almost complex structures (with  $\kappa$  fixed)  $C^\infty$ -close uniformly on  $U$  to the  $\kappa$ -periodic extension of  $J$ . Periodic perturbations of  $H$  can be incorporated into the theorem in a similar fashion.

*Remark 3.3.* It might also be worth pointing out that the condition  $u(\partial\Sigma) \subset \partial U$  should be understood as that the map  $(s, t) \mapsto (t, u(s, t))$  sends  $\partial\Sigma$  to the boundary of the domain in  $S_\kappa^1 \times M$  covering  $U \subset S^1 \times M$ . Furthermore, note that, when dealing with a countable collection of maps  $u$ , we can always ensure that the domains  $\Sigma$  have smooth boundary by slightly shrinking  $U$ .

We apply Theorem 3.1 in the following setting. Let  $u$  be a solution of the Floer equation for  $H^{\natural\kappa}$ , which is asymptotic to  $\gamma^\kappa$  on one side and to some  $\kappa$ -periodic orbit  $x$  on the other. Assume furthermore that  $x$  does not enter a neighborhood  $U$  of  $\gamma$ . Then  $E(u) > c_\infty$  for some constant  $c_\infty > 0$ , which, by the theorem, is independent of  $\kappa$  and  $u$ . Moreover, we can replace  $H^{\natural\kappa}$  by any  $\kappa$ -periodic Hamiltonian  $K$  equal to  $H^{\natural\kappa}$  on  $U$ .

*Remark 3.4.* The assumption that  $\gamma$  is hyperbolic is absolutely crucial in Theorem 3.1. Consider, for instance, the Hamiltonian  $H(z) = a|z|^2$  generating an irrational rotation of  $\mathbb{C} = \mathbb{R}^2$ . Then, as a direct calculation shows, the ball crossing energy can get arbitrary small for arbitrarily large iterations  $\kappa$ . Alternatively, this can be seen by examining the height function  $H$  on  $S^2$  and observing that, for a suitable choice of  $\kappa$  and of the cappings of the polls  $x$  and  $y$ , the difference  $|\mathcal{A}_{H^{\natural\kappa}}(\bar{x}^\kappa) - \mathcal{A}_{H^{\natural\kappa}}(\bar{y}^\kappa)|$  can be arbitrarily small while there are always Floer trajectories connecting  $\bar{x}^\kappa$  and  $\bar{y}^\kappa$ . (The latter fact follows from the structure of the cap or quantum product on  $S^2$ ; see Example 2.1.)

**3.2. Proof of Theorem 3.1.** The key feature of hyperbolic orbits the argument relies on is that, given a closed neighborhood of  $\gamma$ , the orbit of  $\varphi_H^t$  with the initial condition on (or near) the boundary of the neighborhood cannot stay within the neighborhood simultaneously for large positive and large negative times.

To simplify the setting, we first observe that without loss of generality we can assume that  $\gamma(t)$  is a constant orbit, which we still denote by  $\gamma$ . This is a consequence of the fact that there exists a one-periodic loop of Hamiltonian diffeomorphisms  $\eta_t$ , defined on a neighborhood of  $\gamma$ , such that  $\eta_t(\gamma(0)) = \gamma(t)$  (see, e.g., [Gi2, Section 5.1]), which allows us to replace  $\varphi_H^t$  by  $\eta_t^{-1}\varphi_H^t$  and  $H$  by the corresponding Hamiltonian. This step is not really necessary, but it does simplify the notation and the geometrical picture, and, since the loop  $\eta_t$  is in general only local, does not require the orbit  $\gamma$  to be contractible. From now on, we can assume that  $U$  and other neighborhoods of  $\gamma$  are actually subsets of  $M$  rather than of  $S_\kappa^1 \times M$ .

Next, let us fix another closed neighborhood  $B \subset \text{int}(U)$  of  $\gamma$  with smooth boundary. If  $B$  is sufficiently small, there exists a constant  $L_0 > 0$ , depending only on  $B$  and  $H$ , such that for all initial times  $\tau \in [0, 1]$  no integral curve of  $H$  passing through a point of  $\partial B$  (or near  $\partial B$ ) at the moment  $\tau$  can stay in  $B$  for all  $t$  with  $|t - \tau| < L_0$ . This readily follows from the assumptions that  $\gamma$  is hyperbolic and  $H$  is one-periodic in time. (Note that the role of  $\tau$  here is to account for the fact that the vector field  $X_H$  and the flow  $\varphi_H^t$  are time-dependent. If  $\varphi_H^t$  were a true flow, we would be able to take  $\tau = 0$ . In our setting, where  $H$  is one-periodic in time, it suffices to require that  $\tau \in [0, 1]$ .)

The idea of the proof is that if there is a sequence of solutions  $u_i$  of the Floer equation with  $E(u_i) \rightarrow 0$ , this sequence converges to a zero energy solution defined on a domain in  $\mathbb{C}$ . The limit solution is independent of  $s$ , and hence an integral curve of  $H$ . The sequence  $u_i$  can be chosen so that this integral curve is defined on the interval  $[\tau - L, \tau + L]$  for some  $L > L_0$  and  $\tau \in [0, 1]$ , contained in  $B$ , and tangent to  $\partial B$  at the moment  $\tau$ . This is impossible since  $\gamma$  is hyperbolic.

Thus, arguing by contradiction, assume now that there exist a sequence of  $\kappa_i$ -periodic almost complex structures  $J_i$  on  $M$ , compatible with  $\omega$  and  $C^\infty$ -converging to  $J$  uniformly on  $\mathbb{R} \times U$ , and a sequence  $u_i: \Sigma_i \rightarrow U$  of solutions of (2.6) (for  $J_i$  with  $\kappa = \kappa_i$ ), satisfying the hypotheses of Theorem 3.1 and such that  $E(u_i) \rightarrow 0$ . As we show below, we may assume without loss of generality that the maps  $u_i$  and the almost complex structures  $J_i$  have the following properties:

- (i) The domains  $\Sigma_i$  have smooth boundary.
- (ii) The region  $[0, \infty) \times S_{\kappa_i}^1$  is the largest half-cylinder in  $\Sigma_i$  mapped by  $u_i$  into  $B$ , i.e.,  $u_i([0, \infty) \times S_{\kappa_i}^1) \subset B$  and  $u_i(\{0\} \times S_{\kappa_i}^1)$  touches  $\partial B$  at at least one point  $u_i(0, \tau_i)$ , and furthermore  $0 \leq \tau_i \leq 1$ .
- (iii) The sequences  $\tau_i$  and  $u_i(0, \tau_i)$  converge.

Here (i) can be ensured by slightly altering the domain  $U$ ; see Remark 3.3. Furthermore, the first part of (ii) readily follows since  $H$  is independent of  $s$ , and hence (2.6) is translation invariant. (This requires changing  $u_i$  by applying a translation in  $s$ , which clearly does not change the energy.) Next, to ensure that  $0 \leq \tau_i \leq 1$ , we apply an integer translation in  $t$  to  $u_i$  and  $J_i$ . Since the almost complex structures  $J_i$   $C^\infty$ -converge to  $J$  uniformly in  $t \in \mathbb{R}$ , the same is true for the translated almost complex structures. This procedure changes the almost complex structures  $J_i$  and the solutions  $u_i$ , but it does not effect the energy of  $u_i$ , which therefore still goes to zero. Finally, it suffices to pass to a subsequence to establish (iii).

Let us lift the domains  $\Sigma_i$  to the domains  $\hat{\Sigma}_i$  in the universal covering  $\mathbb{C}$  of  $\mathbb{R} \times S^1_{\kappa_i}$  and view the maps  $u_i$  (keeping the notation) as maps  $u_i: \hat{\Sigma}_i \rightarrow M$ , which are  $\kappa_i$ -periodic in  $t$ . As is well-known, the graph  $\hat{u}_i$  of  $u_i$  is a  $\tilde{J}_i$ -holomorphic curve in  $\mathbb{C} \times M$  with respect to an almost complex structure  $\tilde{J}_i$  which incorporates both  $J_i$  and  $X_H$ . Recall that the projection  $\pi: \mathbb{C} \times M \rightarrow \mathbb{C}$  is holomorphic, and hence the projection of  $\hat{u}_i$  to  $\mathbb{C}$  is also a holomorphic map. Set  $\tau = \lim \tau_i \in [0, 1]$ .

Pick arbitrary constants  $L > L_0$  and  $a > 0$  and fix a rectangle

$$\Pi = [-a, a] \times [\tau - L, \tau + L] \subset \mathbb{C}.$$

From now on we will focus on the restrictions  $u_i|_{\Pi}$ , which we still denote by  $u_i$ . Let  $\tilde{u}_i$  be the graph of this restriction, i.e., intersection of  $\hat{u}_i$  with  $\Pi \times U$ . Clearly the boundary of  $\tilde{u}_i$  lies in  $\partial(\Pi \times U)$  and

$$\text{Area}(\tilde{u}_i) \leq \text{Area}(\Pi) + E(u_i) < \text{const},$$

where the constant on the right is independent of  $i$ . Let us now shrink  $\Pi$  and  $U$  slightly. To be more precise, set

$$\Pi' = [-a', a'] \times [\tau - L', \tau + L'] \subset \Pi,$$

where  $0 < a' < a$  and  $L_0 < L' < L$ , and let  $U'$  be a closed neighborhood of  $\gamma$  such that  $B \subset \text{int}(U')$  and  $U' \subset \text{int}(U)$ .

By Fish's version of the Gromov compactness theorem, [Fi, Theorem A], the intersections of the  $\tilde{J}_i$ -holomorphic curves  $\tilde{u}_i$  with  $\Pi' \times U'$  Gromov-converge, after passing if necessary to a subsequence, to a (cusp)  $\tilde{J}$ -holomorphic curve  $\tilde{u}$  in  $\Pi' \times U'$  with boundary in  $\partial(\Pi' \times U')$ . This holomorphic curve is a union of multi-sections over subsets of  $\Pi'$  and possibly some components contained in the fibers of  $\pi$  (the bubbles) with boundary in  $\partial U'$ . The latter are points since  $E(u_i) \rightarrow 0$ . Furthermore,  $\tilde{u}$  is in fact a unique section over some subset of  $\Pi'$ . The reason is that the intersection index of  $\tilde{u}$  with the fiber over a regular point  $(s, t)$  of its projection to  $\Pi'$  is either one or zero – the intersection index of  $\tilde{u}_i$  with the fiber. For instance, the index is one when  $(s, t)$  is in the domain of each  $u_i$  and the distance from  $u_i(s, t)$  to  $\partial U'$  stays bounded away from zero as  $i \rightarrow \infty$ . (Here and below we loosely follow the proof of [McL, Lemma 2.3]. Note also that we need the parameters  $a'$  and  $L'$  and the neighborhood  $U'$  to be “generic”.)

To summarize,  $\tilde{u}$  is the graph of a solution  $u$  of the Floer equation, (2.6), defined on some (obviously connected) subset  $D$  of  $\Pi'$ . Moreover, as is easy to see from [Fi], after making an arbitrarily small change to  $a'$  and  $L'$  we can ensure that the domain  $D$  of  $u$  has piece-wise smooth boundary. The maps  $u_i$  uniformly converge to  $u$  on compact subsets of  $\text{int}(D)$ .

Observe now that  $D$  contains the half-rectangle  $\Pi^+ = \{s > 0\} \cap \Pi'$ . This is a consequence of the fact that each  $\tilde{u}_i$  projects surjectively onto the  $\{s \geq 0\}$ -part of  $\Pi$  or, in other words, this part of  $\Pi$  is in the domain of  $u_i$ . As a consequence,  $D$  also contains the closure of  $\Pi^+$  and, in particular, the point  $(0, \tau)$ . Moreover, this point is in the interior of  $D$  since the distance from the points  $u_i(0, \tau) \in B$  to  $\partial U'$  stays bounded away from zero. Thus

$$u(0, \tau) = p := \lim u_i(0, \tau_i) \in \partial B.$$

Clearly, the solution  $u$  has zero energy. Thus  $\partial_s u(s, t) = 0$  identically on  $D$ , and hence  $u(s, t)$  is an integral curve  $u(t)$  of the Hamiltonian flow of  $H$ . This integral curve passes through the point  $p \in \partial B$  at the moment  $\tau$ , and  $u(t) \in B$  for all

$t \in [\tau - L', \tau + L']$ , which is impossible due to our choice of  $L_0$  and the fact that  $L' > L_0$ . This contradiction completes the proof of the theorem.

*Remark 3.5.* Under some additional assumptions on  $J$  and  $H$  and  $u$  in Theorem 3.1, one can obtain a much more precise lower bound on the energy of  $u$ . Assume, for instance, that  $H$  is a quadratic hyperbolic Hamiltonian on  $\mathbb{R}^{2n}$ , and hence  $\varphi_H^t$  is a linear flow. Furthermore, let us require that  $\Sigma = [0, \infty) \times S_\kappa^1$  and  $u$  be a solution of (2.6) for a linear complex structure  $J$  suitably adapted to  $H$ . Then

$$E(u) > \text{const} \|u(0)\|_{L^2}^2,$$

where  $u(0) = u(0, \cdot)$  and  $\text{const} > 0$  is independent of  $\kappa$  and  $u$ , but depends on  $H$  and  $J$ . (This can be proved by analyzing (2.6) via the Fourier expansion of  $u$  in  $t$ .) In particular, if  $u(\partial\Sigma)$  lies outside the ball of radius  $R$ , i.e.,  $\|u\|_{L^\infty} > R$ , we have  $E(u) > \text{const} \cdot R^2 \kappa$ . As a consequence, under these conditions, the ball-crossing energy  $c_\infty$  grows linearly in  $\kappa$ .

*Remark 3.6.* An argument similar to (and in fact simpler than) the proof of Theorem 3.1 shows that, for a fixed period  $\kappa$ , the ball crossing energy is bounded away from zero by a constant independent of a solution of the Floer equation for any isolated  $\kappa$ -periodic orbit of an arbitrary  $\kappa$ -periodic Hamiltonian; cf. the proof of [McL, Lemma 2.3]. However, it is essential that in this case the period  $\kappa$  is fixed. The bound depends on the neighborhood of the orbit, the Hamiltonian and the almost complex structure, and, in contrast with Theorem 3.1, on  $\kappa$ .

*Remark 3.7.* As is clear from the proof, Theorem 3.1 holds in some instances for periodic orbits which are not necessarily hyperbolic; for instance, the argument applies to  $\gamma \equiv 0$  for the degenerate Hamiltonians  $x^2 - y^4$  and  $x^4 - y^4$  and the “monkey saddle” on  $\mathbb{R}^2$ .

#### 4. PROOF OF THE MAIN THEOREM

**4.1. Idea of the proof.** Fix a neighborhood  $U$  of  $\gamma$  as in Theorem 3.1. To explain the idea of the proof, let us, focusing on Case (i), first show that  $c_\infty \leq \lambda_0$  whenever no other periodic orbit enters  $U$ . Assume the contrary:  $c_\infty > \lambda_0$ . Without loss of generality, we may also assume that the orbit  $\gamma$  is one-periodic and that  $\mathcal{A}_H(\bar{\gamma}) = 0$  for some capping of  $\gamma$ . Then the chain  $\bar{\gamma} \in \text{CF}_*^{(a,b)}(H)$  is closed for any interval  $(a, b)$  containing  $[-\lambda_0, 0]$  and such that  $b - a < c_\infty$ . Moreover,  $[\bar{\gamma}] \neq 0$  in  $\text{HF}_*^{(a,b)}(H)$ . Acting on  $[\bar{\gamma}]$  by  $\alpha$  and applying (2.15) with  $\beta * \alpha = q[M]$ , we see that  $\Phi_\alpha([\bar{\gamma}]) \neq 0$  in  $\text{HF}_*^{(-\lambda_0, 0)}(H)$ . Thus  $\bar{\gamma}$  is connected by a Floer trajectory  $u$  to some orbit  $\bar{y}$  with action in the range  $(-\lambda_0, 0)$ . This is impossible since  $E(u) = |\mathcal{A}_H(\bar{y})| < c_\infty$ .

Now, if we knew that, as in Remark 3.5, the energy  $c_\infty$  goes to infinity as  $\kappa$  grows, we could make  $c_\infty$  greater than  $\lambda_0$  by passing to an iteration. This would prove that there are periodic orbits entering an arbitrarily small neighborhood of  $\gamma$  – an assertion much stronger than the theorem. Of course, we do not know whether or not  $c_\infty \rightarrow \infty$  as  $\kappa \rightarrow \infty$ . In the proof, we circumvent this difficulty by replacing  $H$  with a carefully chosen iteration  $H^{\natural\kappa}$  (using the condition that  $N \geq n/2 + 1$ ) so that the above argument still goes through even when  $c_\infty < \lambda_0$ ; see Lemma 4.1.

**4.2. Proof of Theorem 1.1.** Arguing by contradiction, let us assume that  $\varphi_H$  has finitely many periodic orbits. In this case, by replacing  $H$  with its iteration, we can also assume that  $\gamma$  is a one-periodic orbit and that  $H$  is perfect in the



terminology of [GK]: all periodic orbits of  $\varphi_H$  are fixed points. Furthermore, again by passing if necessary to an iteration, we can guarantee that the mean index of  $\gamma$  (equal to the Conley–Zehnder of  $\gamma$ ) with respect to any capping is divisible by  $2N$ . Then there exists a capping such that the mean index is zero. Let  $\bar{\gamma}$  be the orbit  $\gamma$  equipped with this capping:  $\Delta(\bar{\gamma}) = 0$ . We keep the notation  $H$  for this iterated Hamiltonian and assume it to be one-periodic in time, which can always be achieved by a reparametrization. Finally, by adding a constant to  $H$ , we can ensure that  $\mathcal{A}_H(\bar{\gamma}) = 0$ .

Fix a one-periodic in time almost complex structure  $J^0$ . Let  $U$  be a small closed neighborhood of  $\gamma$  such that no periodic orbit of  $H$  other than  $\gamma$  intersects  $U$ . By Theorem 3.1 applied to  $U$ , there exists a constant  $c_\infty > 0$  such that, for any  $\kappa$ , every non-trivial  $\kappa$ -periodic solution of the Floer equation for the pair  $(H, J^0)$  asymptotic to  $\gamma^\kappa$  as  $s \rightarrow \pm\infty$  has energy greater than  $c_\infty$ .

Denote by  $a_i \in S_{\lambda_0}^1$  the actions of one-periodic orbits of  $H$  taken up to  $\lambda_0$ , and hence independent of capping. Let also  $\tilde{a}_i \in \mathbb{R}$  stand for the augmented actions of one-periodic orbits. Due to the assumption that  $H$  has finitely many fixed points, the collections  $\{a_i\}$  and  $\{\tilde{a}_i\}$  are finite.

Furthermore, fix a large constant  $C > 0$  and a small constant  $\epsilon > 0$ . The values of these constants are to be specified later; see (4.4) and (4.5).

As is easy to show using the Kronecker theorem and (2.5), there exists an integer period  $\kappa > 0$  such that for all  $i$

$$\|\kappa \cdot a_i\|_{\lambda_0} < \epsilon \text{ and either } \tilde{a}_i = 0 \text{ or } |\kappa \cdot \tilde{a}_i| > C. \quad (4.1)$$

Here  $\|a\|_{\lambda_0} \in [0, \lambda_0/2]$  stands for the distance from  $a \in S_{\lambda_0}^1$  to 0.

Let  $K$  be a  $\kappa$ -periodic Hamiltonian sufficiently  $C^2$ -close to  $H^{\natural\kappa}$ . Denote by  $\mathcal{P}$  the collection of contractible  $\kappa$ -periodic orbits of  $K$  and by  $\bar{\mathcal{P}}$  the collection of capped  $\kappa$ -periodic orbits. Then (4.1) readily implies that for every  $x \in \mathcal{P}$  we have

$$\|\mathcal{A}_K(x)\|_{\lambda_0} < \epsilon \text{ and} \quad (4.2)$$

$$\text{either } |\tilde{\mathcal{A}}_K(x)| < \delta \text{ or } |\tilde{\mathcal{A}}_K(x)| > C, \quad (4.3)$$

where  $\delta > 0$ , to be specified later (see (4.5)), can be made arbitrarily small. In particular, by (4.2),  $\mathcal{S}(K)$  is contained in the  $\epsilon$ -neighborhood of  $\lambda_0\mathbb{Z}$ .

On the other hand, assume that  $K$  is non-degenerate (and again  $\kappa$ -periodic) and equal to  $H^{\natural\kappa}$  on  $U$ . (We do not need  $K$  to be  $C^2$ -close to  $H^{\natural\kappa}$ .) Then for any  $\kappa$ -periodic almost complex structure  $J$ , which is sufficiently close to (the  $\kappa$ -periodic extension of)  $J^0$ , all non-trivial  $\kappa$ -periodic solutions of the Floer equation for the pair  $(K, J)$  asymptotic to  $\gamma^\kappa$  as  $s \rightarrow \pm\infty$  have energy greater than  $c_\infty$ . This follows from the “moreover” part of Theorem 3.1 (see Remark 3.2) or, alternatively, can be easily established as a consequence of the compactness theorem for solutions of the Floer equation for a fixed period, see, e.g., [Sa, Corollary 3.4].

Let us now fix a  $\kappa$ -periodic Hamiltonian  $K$  meeting all of the above conditions:  $K$  is non-degenerate, sufficiently  $C^2$ -close to  $H^{\natural\kappa}$ , and equal to  $H^{\natural\kappa}$  on  $U$ . (When  $H^{\natural\kappa}$  is non-degenerate, we can take  $K = H^{\natural\kappa}$ .) Furthermore, fix a  $\kappa$ -periodic almost complex structure  $J$  such that the regularity requirements for the pair  $(K, J)$  are satisfied, and every non-trivial solution of the Floer equation for  $(K, J)$  asymptotic to  $\gamma^\kappa$  as  $s \rightarrow \pm\infty$  has energy greater than  $c_\infty$ . In particular, as a consequence of the regularity, the filtered Floer complex  $\text{CF}_*(K)$  for the pair  $(K, J)$  is defined.

Before proceeding with the proof, let us spell out, in the logical order, the sequence of choices made above. Once  $H$  has been made perfect, we start by fixing  $U$  and an almost complex structure  $J^0$ . These choices determine  $c_\infty$ . Next we choose a large constant  $C > 0$  and small constants  $\epsilon > 0$  and  $\delta > 0$ ; we will explicitly specify our requirements on these constants shortly. Then we pick  $\kappa$  to satisfy (4.1), and then the Hamiltonian  $K$  meeting several conditions including (4.2) and (4.3). Finally, we fix the almost complex structure  $J$ .

In the rest of the proof, the exact restrictions on the constants  $C$ ,  $\epsilon$  and  $\delta$  are essential. The constant  $C$  is chosen so that

$$C > 2n\lambda + \nu\lambda_0. \quad (4.4)$$

The constants  $\epsilon$  and  $\delta$  are positive, and

$$\epsilon < c_\infty \text{ and } 2(\epsilon + \delta) < \lambda. \quad (4.5)$$

Note that  $\kappa$  depends on  $\epsilon$  (and of course  $C$ ), which in turn depends on the choice of  $c_\infty$  and ultimately on the choice of  $U$ .

From now on, we work exclusively with the Hamiltonian  $K$ , its  $\kappa$ -periodic orbits, and the Floer equation for  $(K, J)$ . To simplify the notation, we write  $\gamma$  and  $\bar{\gamma}$  in place of  $\gamma^\kappa$  and  $\bar{\gamma}^\kappa$ . Our ultimate goal is to show that there exists an orbit  $\bar{\gamma}$  of  $K$  with action not in the  $\epsilon$ -neighborhood of  $\lambda_0\mathbb{Z}$ , which is impossible by (4.2). This will complete the proof of the theorem.

**Lemma 4.1.** *Let  $C' = C - \lambda(n+1)/2$ . The orbit  $\bar{\gamma}$  is not connected by a solution of the Floer equation of relative index  $\pm 1$  to any  $\bar{x} \in \bar{\mathcal{P}}$  with action in  $(-C', C')$ . In particular,  $\bar{\gamma}$  is closed in  $\text{CF}_*^{(-C', C')}(K)$  and  $[\bar{\gamma}] \neq 0$  in  $\text{HF}_*^{(-C', C')}(K)$ . Moreover,  $\bar{\gamma}$  must enter every cycle representing  $[\bar{\gamma}]$  in  $\text{HF}_*^{(-C', C')}(K)$ .*

*Proof.* Arguing by contradiction, assume that  $\bar{x}$  is connected to  $\bar{\gamma}$  by a solution  $u$  of the Floer equation of relative index  $\pm 1$ . Thus  $\mu_{\text{CZ}}(\bar{x}) = \pm 1$ , and hence  $|\Delta_K(\bar{x})| < n+1$  by (2.2) and also  $x \neq \gamma$ . By (4.3), either  $|\tilde{\mathcal{A}}_K(x)| < \delta$  or  $|\tilde{\mathcal{A}}_K(x)| > C$ .

Let us first consider the former case:  $|\tilde{\mathcal{A}}_K(x)| < \delta$ . We have  $E(u) > c_\infty > \epsilon$ , and therefore

$$|\mathcal{A}_K(\bar{x})| = E(u) > \lambda_0 - \epsilon,$$

by (4.2) and (4.5). It follows from the condition  $|\tilde{\mathcal{A}}_K(x)| < \delta$  and the second inequality in (4.5) that

$$|\Delta_K(\bar{x})| > \frac{2}{\lambda}(\lambda_0 - \epsilon - \delta) = 2N - \frac{2(\epsilon + \delta)}{\lambda} > 2N - 1.$$

Thus, using the requirement that  $N \geq n/2 + 1$ , we have

$$|\mu_{\text{CZ}}(\bar{x})| > 2N - 1 - n \geq n + 2 - 1 - n = 1,$$

which is impossible since  $\mu_{\text{CZ}}(\bar{x}) = \pm 1$ .

In the latter case,  $|\tilde{\mathcal{A}}_K(x)| > C$ , we have

$$|\mathcal{A}_K(\bar{x})| > C - \frac{\lambda}{2}|\Delta_K(\bar{x})| > C - \frac{\lambda}{2}(n+1) = C',$$

where we used (2.2) in the last inequality. Hence the orbit  $\bar{x}$  is outside the action range  $(-C', C')$ .  $\square$

Since the Floer differential commutes with recapping, the same is true for the orbit  $q^\nu \bar{\gamma} = \bar{\gamma} \# (\nu A_0)$  with the shifted action range  $(-C', C') - \nu \lambda_0$ .

Let now  $(a, b)$  be any open interval containing  $[-\nu \lambda_0, 0]$  and contained in the intersection of the intervals  $(-C', C')$  and  $(-C', C') - \nu \lambda_0$ . Such an interval exists since  $-C' < -\lambda_0$  and  $C' - \nu \lambda_0 > 0$  due to our choice of  $C$ ; see (4.4). Then the above assertions hold for both capped orbits  $\bar{\gamma}$  and  $q^\nu \bar{\gamma}$  and the interval  $(a, b)$ : these orbits are not connected by a Floer trajectory of relative index  $\pm 1$  to any capped orbit with action in this interval; the chains  $\bar{\gamma}$  and  $q^\nu \bar{\gamma}$  are closed in  $\text{CF}_*^{(a, b)}(K)$ ; the capped orbits must enter any representatives of the classes  $[\bar{\gamma}]$  and, respectively,  $q^\nu [\bar{\gamma}]$  in  $\text{HF}_*^{(a, b)}(K)$ ; and, in particular, these classes are both non-zero.

**Lemma 4.2.** *The Hamiltonian  $K$  has a periodic orbit with action outside the  $\epsilon$ -neighborhood of  $\lambda_0 \mathbb{Z}$ .*

The main theorem follows from this lemma. For the existence of such an orbit contradicts (4.2).

*Proof.* Applying (2.16) and (2.15) to  $[\bar{\gamma}]$  with  $(c, c') = (a, b)$ , we see that the image (under the quotient-inclusion map) in  $\text{HF}_*^{(a, b)}(K)$  of the class  $\Phi_\alpha([\bar{\gamma}])$  is non-zero; for  $q^\nu [\bar{\gamma}] \neq 0$  in this homology group.

Recall that  $\alpha$  and  $\beta$  are both ordinary homology classes and let, as in Section 2.3,  $\sigma$  and  $\eta$  be generic cycles representing  $\alpha$  and, respectively,  $\beta$ . Then the chain  $\Phi_\sigma(\bar{\gamma})$  in  $\text{CF}_*^{(a, b)}(K)$  is also non-zero. Moreover, the chain  $\Phi_\eta \Phi_\sigma(\bar{\gamma})$  represents the class  $q^\nu [\bar{\gamma}]$ , and hence the orbit  $q^\nu \bar{\gamma}$  enters this chain. As a consequence, there exists an orbit  $\bar{y}$  in the chain  $\Phi_\sigma(\bar{\gamma})$  which is connected to both  $\bar{\gamma}$  and  $q^\nu \bar{\gamma}$  by Floer trajectories.

Therefore,

$$-\epsilon > \mathcal{A}_K(\bar{y}) > -\nu \lambda_0 + \epsilon. \quad (4.6)$$

When  $\nu = 1$ , (4.6) turns into  $-\epsilon > \mathcal{A}_K(\bar{y}) > -\lambda_0 + \epsilon$ , which concludes the proof of the lemma in Case (i).

To establish the lemma in Case (ii), we first observe that  $\mu_{\text{CZ}}(\bar{y}) = |\alpha| - 2n$ , and thus  $|\Delta_K(\bar{y})| < 4n$ . (In fact,  $-3n < \Delta_K(\bar{y}) < n - 1$ .) Next, as in the proof of Lemma 4.1, we consider two sub-cases:  $|\tilde{\mathcal{A}}_K(\bar{y})| < \delta$  and  $|\tilde{\mathcal{A}}_K(\bar{y})| > C$ .

In the former case, we argue by contradiction. Assume that  $\mathcal{A}_K(\bar{y})$  is in the  $\epsilon$ -neighborhood of  $\lambda_0 \mathbb{Z}$ . Then, by (4.6),  $\mathcal{A}_K(\bar{y}) < -\lambda_0 + \epsilon$ , and hence we obtain using (4.5) that

$$\Delta_K(\bar{y}) < \frac{2}{\lambda}(-\lambda_0 + \epsilon + \delta) = -2N + \frac{2(\epsilon + \delta)}{\lambda} < -2N + 1.$$

Therefore,

$$|\alpha| - 2n = \mu_{\text{CZ}}(\bar{y}) < -2N + 1 + n$$

or, in other words,

$$|\alpha| < 3n + 1 - 2N,$$

which contradicts the assumption of the theorem that  $|\alpha| \geq 3n + 1 - 2N$ .

In the latter case where  $|\tilde{\mathcal{A}}_K(\bar{y})| > C$ , we have

$$|\mathcal{A}_K(\bar{y})| > C - \frac{\lambda}{2} |\Delta_K(\bar{y})| > C - 2n\lambda > \nu \lambda_0$$

by (4.4), which is impossible due to (4.6).  $\square$

This concludes the proof of the theorem.

*Remark 4.3.* It is not hard to see from the proof of Theorem 1.1 that in Case (i) the condition that  $\alpha$  and  $\beta$  are ordinary homology classes can be slightly relaxed and replaced, for instance, by the requirement that  $I_\omega(\alpha) = 0$  and  $I_\omega(\beta) \geq -\lambda_0$ .

*Remark 4.4* (Proof in the case of non-contractible orbits). When the orbit  $\gamma$  is non-contractible, the above argument goes through essentially word-for-word, requiring only minimal modifications (in fact, simplifications) in the beginning of the proof. Indeed, let  $\zeta$  be the free homotopy class of  $\gamma$ . Without loss of generality, we may assume that  $\kappa\zeta \neq 0$  for all positive integers  $\kappa$  or, alternatively, use Remark 2.2. By passing if necessary to the second iteration, we can also ensure that  $d\varphi_H$  at  $T_{\gamma(0)}M$  has an even number of real eigenvalues in  $(-1, 0)$ . Then there exists a trivialization of  $TM$  along  $\gamma$  such that  $\Delta_H(\gamma) = 0 = \mu_{CZ}(\gamma)$ . We take  $z = \gamma$  with this trivialization as a reference loop. In the notation and conventions of Section 2.4, the rest of the proof remains unchanged.

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